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# Analytical solution of unsteady heat conduction in a two-layered material in imperfect contact subjected to a moving heat source

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ABSTRACT

An analytical approach of transient heat conduction in two-layered material, of finite depth, with an imperfect thermal contact, subjected to a moving gaussian laser beam was developed.

The method consists of deriving the solution of the homogeneous part of the heat equation by using the well known separation of variables method and expressing the source term in series form. The porous aspect of granular coating layer on substrate was also taken into account earlier in this modelling work. This model has been successfully applied on a practical system; laser cladding of electronic copper tracks on alumina substrates. This analytical model can be used also for estimation of the thermal contact resistance between layers.



## 1. Introduction

Modelling physical problems dealing with heat transfer occurring in case of scanning sources are of great interest in many technological applications like heat treatments, welding, annealing, … The surface treatment is one among the very important areas where analytical approaches of transient heat transfer are needed, especially in the case of composite materials.

Many practical cases of heat conduction encountered in the surface treatment field concern two or more than two layers (deposit–substrate systems, …), involving a coating material used in order to improve performances of the global medium. In fact, in this kind of problems, it is difficult to get the full closed form solution due to some complexity of the system; like interfacial conditions, and restriction to work in finite dimensions (thin films or layers). Considerable work, dealing with analytical [1–6] and semi-analytical [7–9] modelling of welding, cutting, drilling, etc. is found in the literature, but almost all of them deal with a homogeneous material. Most of the studies done on heat transfer for composite mediums treated the problem numerically, and the papers that deal with this kind of problems analytically; over simplify the system by the assumptions of: semi-infinite domain (no free convection on the boundaries, perfect contact at the interface…). Some studies were done in this field where the authors Adawi et al. [10] obtained a 1D analytical solution of heat conduction in two-layered material by using Laplace transform; under the assumption of perfect thermal contact, fixed laser source, and semi-infinite substrate. The heat losses by convection in the outer boundaries were neglected, which implies that the solution can be valid for very short treatment times, because the convective mode plays an important role if the treatment duration is of the order of few seconds. Haji et al. [11] studied heat conduction in composite materials of finite dimensions by using Green's function. Baïri et al. [12] studied the effect of a coating on the thermal behaviour of a body subjected to multiple moving heat sources. Some other authors studied the heat transfer within thermal contact resistance [13,14]. Couëdel et al. [15], derived a 2D-analytical model to calculate thermal cycles in regions of limited width using a vibrated and non-vibrated moving thermal source. Alilat et al. [16] studied the 3D thermal behaviour of a rotating disc subjected to an excentric circular heat source. Some configurations of moving or variable heat sources have been studied from analytical solution [17]. When phenomena are fast and transient, some precautions are needed to carry out the numerical computation [16–19].

De Monte has studied the heat conduction in homogeneous materials (without source term) for 1D and 2D using separation of variables method, for perfect thermal contact [20–22]. But while treating imperfect thermal contacts [24] the author simplified the system with the assumptions of steady state and similar thermal

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**Nomenclature**

| | |
|---|---|
| $a_i$ | dimension along $y$ [m] |
| $A_i$ | laser absorptivity |
| $Cp_i$ | heat capacity [J kg$^{-1}$ K$^{-1}$] |
| $f_i(y)$ | initial temperature difference [K] |
| $G_i(t)$ | temporal eigenfunction |
| $h_i$ | heat transfer coefficient [W m$^{-2}$ K$^{-1}$], Boundaries ($y = a_1$ and $y = -a_2$). |
| $k_i$ | thermal conductivity [W m$^{-1}$ K$^{-1}$] |
| $Nu$ | Nusselt number |
| $Pr$ | Prandtl number |
| $Gr$ | Grashof number |
| $P$ | laser power density [W m$^{-2}$] |
| $P_0$ | laser power [W] |
| $q_i$ | volumetric source term [W m$^{-3}$] |
| $r_0$ | radius of the Gaussian laser beam [m] |
| $t$ | time [s] |
| $T_{amb}$ | surrounding fluid temperature [K] |
| $T_{i0}$ | initial temperature [K] |
| TCR | thermal contact resistance [K m$^{-2}$ W$^{-1}$] |
| $v$ | velocity [m s$^{-1}$] |
| $y$ | space coordinate in $y$ direction |
| $Y_{i,n}$ | the $n$th space eigenfunction [m] |
| $Y_{id,n}$ | the $n$th dimensionless space eigenfunction |

*Greek symbols*

| | |
|---|---|
| $\alpha_i$ | thermal diffusivity [m$^2$ s$^{-1}$] |
| $\lambda_{i,n}$ | eigenvalues along $y$ [m$^{-1}$] |
| $\epsilon_i$ | transmission coefficient |
| $\varphi_n$ | temporal function |
| $\Phi_n$ | temporal function |
| $\varphi$ | porosity (%) |
| $\rho_i$ | density [kg m$^{-3}$] |
| $\theta_i$ | $\theta_i = T_{amb} - T_i$ [K] |
| $\tau$ | time duration [s] |

*Subscripts*

| | |
|---|---|
| H | homogeneous |
| $i$ | 1, 2 |
| m | mixture |
| $n$ | integer parameter |
| f | fluid phase |
| s | solid phase |

diffusivity for the two materials. Salt [23] derived an analytical solution for classical boundary conditions but without source term for perfect thermal contact. A few analytical solutions can be found in books of Ozisik and Mikhailov [24,25], Jaeger and Carslaw [26]. But they give analytical solutions for simplified boundary conditions.

That is why this work is focused on two-layered materials with an imperfect thermal contact, under a moving heat source. The proposed model can be used for verification of numerical codes if experimental results are unavailable.

Another aspect studied here is the porosity of the deposit layer, so the first layer is considered as thin layer made of powder of granular particles. Theoretical and applied research in flow, heat and mass transfer in porous media have received increased attention during the past decades [27–31]. This is due to the importance of this research area in many engineering applications. Significant advances have been made in fluid flow modelling, heat and mass transfer through a porous medium including clarification of several important physical phenomena. Important topics that have received significant interest include porosity variation, thermal dispersion, effects of local thermal equilibrium between the fluid and the solid phase, partially filled porous configurations, and anisotropic porous media, among others.

## 2. Model and solution procedure

### 2.1. Physical problem statement

The aim of this paper is to derive an analytical solution of heat conduction in two-layered material using a new approach, by treating the homogeneous problem first and then expressing the source term separately, this analytical solution takes into account several aspects (Fig. 1).

The extracted solution will consider the imperfect thermal contact at the interface and the gaussian profile (namely a laser beam $TEM_{00}$) moving at a scanning velocity (Fig. 2), and transmitting a portion of the incident energy directly to the substrate (Fig. 3).

In addition, a porosity aspect of the granular deposit was also considered in the present model (see Section 2.4). This modelling work should give an estimation of the thermal field in the composite material (deposit/substrate), for many practical cases such as: design of tracks of copper (powder) on alumina substrate. Ceramic materials were chosen as heat dissipaters owing to their relatively high thermal conductivity. This model can be used also to estimate the TCR value characterizing a deposited splat on a substrate, by comparing experimental results to computations [32,33]. The developed model can also serve for validating numerical development before extension to complex coupling (rapid solidification, nucleation, …) [35,36].

The geometry of the studied physical problem was simplified as sketched above (Fig. 1). In order to simplify the calculation, the interface line was chosen to coincide with $y = 0$. A modified temperature variable have been done $\theta_i = T_{amb} - T_i$ ($i = 1$ for the first layer, $i = 2$ for the second one). Both materials were assumed isotropic with constant thermophysical properties. The surrounding

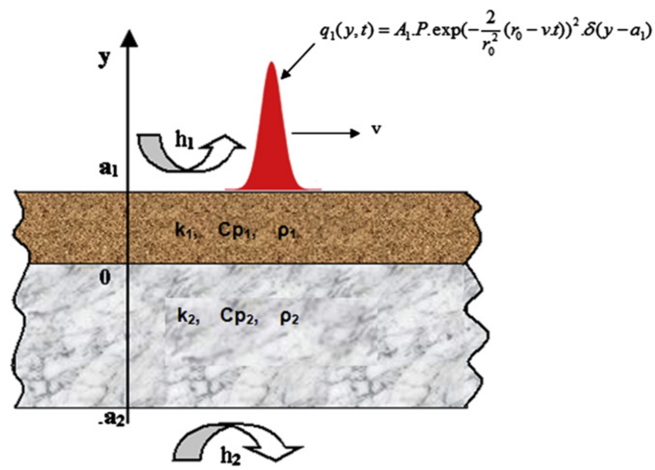


**Fig. 1.** Simplified sketch of the physical problem.

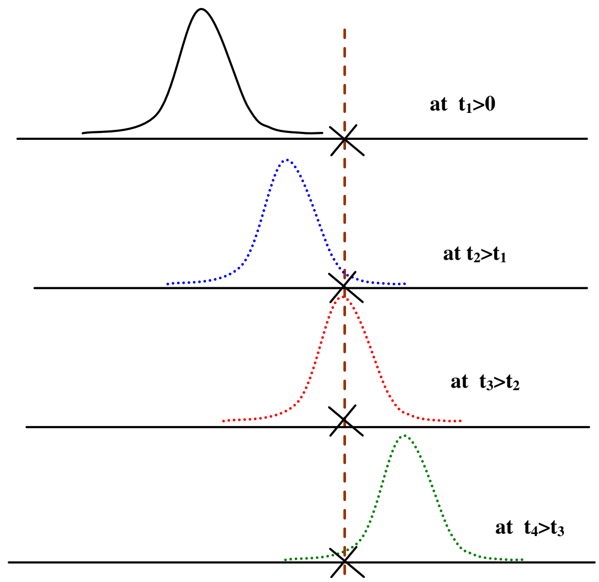


**Fig. 2.** The principle of temperature evolution in a surface spot, under a moving gaussian laser beam.

fluid was assumed uniform at a constant temperature. The exchange coefficients, and TCR along the interface, are considered constant. The phase change and the heat losses by radiation are not taken into account in this study.

It is common to express the laser absorption coefficient as $A = 1 - R$ for metals (dense metals), where $R$ is the reflectivity, this assumption depends on many parameters such as the layer thickness, the laser's wavelength … So under such conditions, a thin layer behaves like a semi-transparent material. Consequently, in the case of thin layers, a portion of the incident laser beam is reflected, $R_1 \times q_1$, the other portion is absorbed by the first layer and converted to heat $A_1 \cdot q_1$, and $(1 - A_1 - R_1)$. $q_1$ is transmitted to the second layer, which follows the same process.

### 2.2. Mathematical formulation

With the assumptions and simplifications above, the governing equation of heat conduction in both layers yields to:

$$\frac{1}{\alpha_i}\cdot\frac{\partial\theta_i}{\partial t} = \frac{\partial^2\theta_i}{\partial y^2} + \frac{q_i}{k_i} \tag{1}$$

where $\alpha_i$, $k_i$ and $q_i$ are respectively: thermal diffusivity, thermal conductivity and heat generation in both layers.

where

$$q_1(y,t) = A_1 \cdot P \cdot \exp\left(-\frac{2}{r_0^2}(r_0 - vt)^2\right)\cdot\delta(y - a_1),$$
$$\text{under the condition } 0 \le y \le a_1,$$
$$t > 0 \text{ and } p = \frac{2\cdot P_0}{\pi\cdot r_0^2} \tag{2-a}$$

and

$$q_2(y,t) = A_2\cdot(1-\varepsilon_1)\cdot P\cdot\exp\left(-\frac{2}{r_0^2}(r_0 - vt)^2\right)\cdot\delta(y - 0),$$
$$\text{under the condition } -a_2 \le y \le 0,\ \ t > 0 \tag{2-b}$$

$a_1$ and $a_2$ are the thicknesses, $A_1$ and $A_2$ are the respective absorptivity coefficients, of the first and second layer respectively.

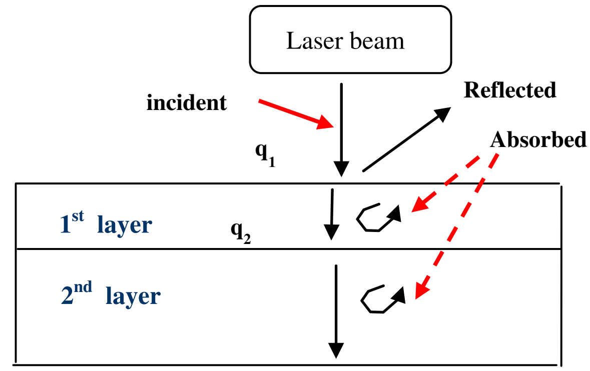


**Fig. 3.** Laser energy transmission to a two-layered material.

$r_0$ and $v$ are respectively, radius and velocity of the laser beam, and $\delta$ is Dirac delta function.

The outer boundary conditions are given by:

$$\pm k_i\left(\frac{\partial\theta_i}{\partial y}\right)_{y=\pm a_i} + h_i\theta_i(y = \pm a_i, t) = 0, \tag{3}$$

where the sign $(-)$ is only valid for $i = 2$.

The inner boundary conditions are given by:

$$k_1\cdot\left(\frac{\partial\theta_1}{\partial y}\right)_{y=0} = k_2\cdot\left(\frac{\partial\theta_2}{\partial y}\right)_{y=0} \tag{4}$$

$$\theta_2(y = 0, t) - \theta_1(y = 0, t) = \text{TCR}\cdot k_1\cdot\left(\frac{\partial\theta_1}{\partial y}\right)_{y=0} \tag{5}$$

where TCR is the thermal contact resistance at the interface.

The initial conditions on both layers are given by:

$$\theta_i(y, t = 0) = f_i(y), \text{ where } f_i(y) = T_{amb} - T_i(y, t = 0)$$
$$= T_{amb} - T_{i0}, \tag{6}$$

### 2.3. Derivation of an analytical solution of the heterogeneous problem

The analytical solution was derived by the method of separation of variables, in two steps: first by using the homogeneous part of the heat equation [20], with the addition of the thermal contact resistance at the interface, and secondly treating the energy source terms in series form. This method is appropriate for linear partial differential equations for finite dimensional medium with constant thermal properties. This method is similar to Green's functions method, but it is easier for composite media.

Step 1 without source terms

The eigenvalues and the eigenfunctions are computed, from the homogeneous heat equation ($q_i = 0$). The homogeneous solution can be expressed as the product of two functions, dependent only on $y$ and $t$.

$$\theta_{Hi}(y,t) = Y_{Hi}(y)\cdot G_{Hi}(t) \tag{7-a}$$

the product formula (7-a) yields two simple ordinary differential equations.

$$\frac{d^2Y_{Hi}(y)}{dy^2} + \lambda_i^2\cdot Y_{Hi}(y) = 0 \tag{7-b}$$

$$\frac{dG_{Hi}(t)}{dt} + \lambda_i^2 \cdot \alpha_i \cdot G_{Hi}(t) = 0 \tag{7-c}$$

and then the functions $Y_{Hi}(y)$ and $G_{Hi}(t)$ are:

$$Y_{Hi}(y) = A_i \cdot \cos(\lambda_i \cdot y) + B_i \cdot \sin(\lambda_i \cdot y) \tag{8-a}$$

$$G_{Hi}(t) = \exp\left(-\lambda_i^2 \cdot \alpha_i \cdot t\right) \tag{8-b}$$

after applying the initial and boundary conditions we obtain :

$$A_i = \mp B_i \cdot R_i(\lambda_i) \text{ the sign } (-) \text{ is valid for } i = 1, \text{ where } R_i(\lambda_i) = \frac{k_i \cdot \lambda_i + h_i \cdot \tan(\lambda_i . a_i)}{h_i - k_i \cdot \lambda_i \cdot \tan(\lambda_i . a_i)}, \frac{\lambda_1^2}{\lambda_2^2} = \frac{\alpha_2}{\alpha_1} \tag{9}$$

And $k_1 \cdot \lambda_1 \cdot B_1 = k_2 \cdot \lambda_2 \cdot B_2$ the eigenvalues ($\lambda_i$, $n$) are the positive real roots other than zero of the following equation:

$$R_1(\lambda_1) + \left(\frac{k_1 \cdot \lambda_1}{k_2 \cdot \lambda_2}\right) \cdot R_2(\lambda_2) + (\lambda_1 \cdot k_1 \cdot \text{TCR}) = 0 \tag{10}$$

in homogeneous case, temperature expressions in both layers are:

$$\theta_{1H}(y,t) = \sum_{n=1}^{\infty} C_n \cdot \exp\left(-\lambda_{1,n}^2 \cdot \alpha_1 \cdot t\right) \cdot Y_{1d,n}(y) \tag{11}$$

$$\theta_{2H}(y,t) = \left(\frac{k_1}{k_2}\right) \cdot \left(\sqrt{\frac{\alpha_2}{\alpha_1}}\right) \cdot \sum_{n=1}^{\infty} C_n \times \left[\exp\left(-\lambda_{1,n}^2 \cdot \alpha_1 \cdot t\right) \cdot Y_{2d,n}(y)\right] \tag{12}$$

where the dimensionless eigenfunctions are given by:

$$Y_{1d,n}(y) = \left(\sin(\lambda_{1,n} \cdot y) - R_{1,n}(\lambda_{1,n}) \cdot \cos(\lambda_{1,n} \cdot y)\right) \tag{13-a}$$

$$Y_{2d,n}(y) = \left(\sin(\lambda_{2,n} \cdot y) + R_{2,n}(\lambda_{2,n}) \cdot \cos(\lambda_{2,n} \cdot y)\right) \tag{13-b}$$

where $R_{i,n}$ and $\lambda_{i,n}$ are calculated from equations (9) and (10).

Step 2 with a source term

The non-homogeneous terms (volumetric source terms) are expressed as a linear combination of the eigenfunctions; which can be demonstrated easily by comparison with the solution obtained using Green's functions method. While both homogenous and non-homogeneous cases of heat transfer can be expressed in series form, so it can be proven that the source terms can be expressed in the same way.

$$q_1(y,t) = \sum_{n=1}^{\infty} \phi_n(t) \cdot Y_{1d,n}(y) \tag{14-a}$$

$$q_2(y,t) = \left(\frac{k_1}{k_2}\right) \cdot \left(\sqrt{\frac{\alpha_2}{\alpha_1}}\right) \cdot \sum_{n=1}^{\infty} \phi_n(t) \cdot Y_{2d,n}(y) \tag{14-b}$$

multiplying side by side by the corresponding eigenfuctions, and integrating over each layer, (eqs. (14-a) and (14-b)) lead to:

$$\int_0^{a_1} k_2 \cdot q_1(y,t) \cdot Y_{1d,n}(y) \cdot dy = \int_0^{a_1} \left(\sum_{n=1}^{\infty} \phi_n(t) \cdot k_2 \cdot Y_{1d,n}^2(y)\right) \cdot dy \tag{15-a}$$

$$\int_{-a_2}^{0} \left(\frac{k_1}{D} \cdot q_2(y,t) \cdot Y_{2d,n}(y)\right) \cdot dy = \int_{-a_2}^{0} \left(\sum_{n=1}^{\infty} \phi_n(t) \cdot k_2 \cdot Y_{1d,n}^2(y)\right) \cdot dy \tag{15-b}$$

where $D = (k_1/k_2) \cdot (\sqrt{\alpha_2/\alpha_1})$

Coefficients $N_n$ are given by using the following orthogonality relation:

$$N_n = k_2 \cdot \int_0^{a_1} Y_{1d,n}(y)^2 \cdot dy + k_1 \cdot \int_{-a_2}^{0} Y_{2d,n}(y)^2 \cdot dy \tag{16}$$

adding (eqs.(15-a) and (15-b)), and using (eq.(16)), one obtain

$$\phi_n(t) = \frac{1}{N_n} \cdot \left(\int_0^{a_1} k_2 \cdot q_1(y,t) \cdot Y_{1d,n}(y) \cdot dy + \int_{-a_2}^{0} \left(\frac{k_1}{D} \cdot q_2(y,t) \cdot Y_{2d,n}(y)\right) \cdot dy\right) \tag{17}$$

then the final expressions of temperatures in both domains can be given by:

$$\theta_1(y,t) = \sum_{n=1}^{\infty} \Phi_n(t) \cdot Y_{1d,n}(y) \tag{18-a}$$

$$\theta_2(y,t) = \left(\frac{k_1}{k_2}\right) \cdot \left(\sqrt{\frac{\alpha_2}{\alpha_1}}\right) \cdot \sum_{n=1}^{\infty} \Phi_n(t) \cdot Y_{2d,n}(y) \tag{18-b}$$

where $\Phi_n(t)$ is determined from the following equation.

Substituting (eqs.(14-a), (14-b), (18-a) and (18-b)) in (eq. (1)) and combining them we obtain:

$$\frac{d\Phi_n(t)}{dt} + \frac{\left(\lambda_{1,n}^2 \cdot \alpha_1 + \lambda_{2,n}^2 \cdot \alpha_2\right)}{2} \cdot \Phi_n(t) = \left(\frac{\alpha_1}{k_1} + \frac{\alpha_2}{k_2}\right) \cdot \phi_n(t) \tag{19}$$

Using the initial conditions (eq. (6)) and solving (eq. (19)) by Laplace transform.

When $\theta_i(y, t = 0) = f_i(y) = 0$, we obtain:

$$\Phi_n(t) = \int_0^1 exp\left(-\left(\frac{(\lambda_{1,n}^2 \bullet \alpha + \lambda_{2,n}^2 \bullet \alpha_2)}{2}\right) \bullet (t - \tau)\right) \bullet \left(\frac{\alpha_1}{k_1} + \frac{\alpha_2}{k_2}\right) \bullet \phi_n(\tau) \bullet d\tau \tag{20}$$

By inserting $\Phi_n(t)$ expression from (eq. 20) in (eqs.(18-a) and (18-b)), Thus the final temperatures can be expressed as:

$$T_i(y,t) = T_{amb} - \theta_i(y,t) \tag{21}$$

## 2.4. Extended solution to a porous layer

Sometimes one of the two materials is a porous medium in many engineering applications that is why it seems interesting to extend the derived analytical solution for conditions involving porous materials.

The method consists of expressing the equations of the mixture (powder and air), the effective thermophysical properties are calculated, which yields mathematically to a two-layered material detailed above.

This model for porous materials is equally applicable on powders. The formulation of the energy conservation in this layer is expressed as following [34].

$$(1-\varphi)\cdot(\rho\cdot C_p)_{1s}\cdot\frac{\partial T_{1s}}{\partial t} = (1-\varphi)\cdot\nabla\cdot(k_{1s}\cdot\nabla T_{1s}) + (1-\varphi)\cdot q_{1s} \quad (22\text{-a})$$

$$(\rho\cdot C_p)_{1f}\cdot\left(\varphi\cdot\frac{\partial T_{1f}}{\partial t} + v_f\cdot\nabla T_{1f}\right) = \varphi\cdot\nabla\cdot\left(k_{1f}\cdot\nabla T_{1f}\right) + \varphi\cdot q_{1f} \quad (22\text{-b})$$

Setting $T_{1s} = T_{1f} = T_1$ and adding, the two equations above yield to:

$$(\rho\cdot C_p)_m\cdot\frac{\partial T_1}{\partial t} + (\rho\cdot C_p)_f\cdot v_f\cdot\nabla T_1 = \nabla\cdot(k_m\cdot\nabla T) + q_{1m} \quad (23)$$

where

$$(\rho\cdot C_p)_m = (1-\varphi)\cdot(\rho\cdot C_p)_{1s} + \varphi\cdot(\rho\cdot C_p)_{1f} \quad (24\text{-a})$$

$$q_m = (1-\varphi)\cdot q_{1s} + \varphi\cdot q_{1f} \quad (24\text{-b})$$

$$k_m = (1-\varphi)\cdot k_{1s} + \varphi\cdot k_{1f} \quad (24\text{-c})$$

In the case where the velocity of the fluid $v_f$ is negligible, then (eq. (23)) becomes:

$$(\rho\cdot C_p)_m\cdot\frac{\partial T_1}{\partial t} = \nabla\cdot(k_m\cdot\nabla T_1) + q_{1m} \quad (25)$$

And then the methodology above can be applicable.

The subscript m refers to the mixture, f to the fluid or the air present in the pores and s to the solid.

## 3. Applications of the model on real problems

The derived model can be applied for a large number of engineering systems involving heterogeneous media subjected to energy sources. Analytical treatment of two-layer material under influence of a heat source is complex as most of the time it involves further different parameters like thermal anisotropy, the contact quality, the heat losses, … Therefore, to justify the validity of this model, it was applied to some elementary applications. The results given by this model were compared to the results given by already present numerical models (Fig. 6), for the following conditions: copper (dense copper) /alumina, a $TEM_{00}$ laser beam, $A_1 = 20\%$, $h_1 = 20\ W\,m^{-2}\,K^{-1}$, $RTC = 10^{-6}\ m^2\,K\,W^{-1}$, $P = 30\ W$, $r_0 = 1\ mm$, $V = 1\ mm\,s^{-1}$.

### 3.1. Some operating parameters estimate

The application treated here is the fabrication of electronic tracks from a thin layer of copper powder deposited on alumina substrate by a laser treatment. So a two-layer rectangular region ($-a_2 \le y \le a_1$; see Fig. 4), initially at uniform temperature $T_0 = T_{amb}$, for times $t > 0$, the convective heat exchange coefficient $h_1$ in the upper surface, $y = a_1$, is estimated below. The other boundary was considered adiabatic ($h_2 = 0$).

The number of the eigenvalues, necessary to reach convergence of the series is 20 first values. Those eigenvalues were computed

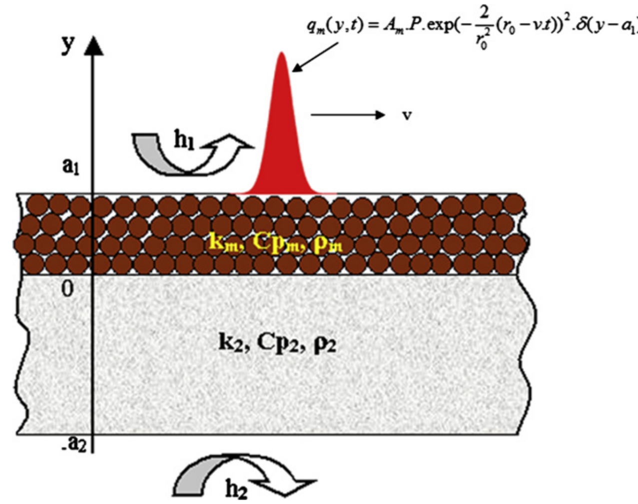


**Fig. 4.** Simplified sketch of the physical problem in the case of porous granular deposit.

numerically using the Newton–Raphson method, which is accurate in this case.

In order to solve this problem we needed to estimate different variables like effective thermal conductivity, thermal contact resistance and heat transfer coefficients.

*Effective thermal conductivity estimate*: in this kind of porous media, the effective thermal conductivity depends on the solid phase (grains) and fluid's one (air); many models can be found in the literature [37–39].

In order to respect the assumption made about the isotropy of both layers, a 5 μm size powder deposit (grains of the same size) was used. For calculation of the effective thermal conductivity, we obtain, by using different models (Fig. 5), for $\varphi = 30\%$ by using the linear relation above (eq. (24-c)) $k_m = 252\ W\,m^{-2}\,K^{-1}$, by using Kanan et al. model [32], $k_m = 224\ W\,m^{-2}\,K^{-1}$ and $k_m = 219\ W\,m^{-2}\,K^{-1}$ by using Wimmer's model [38]. The thermophysical properties were averaged over their temperature range of use.

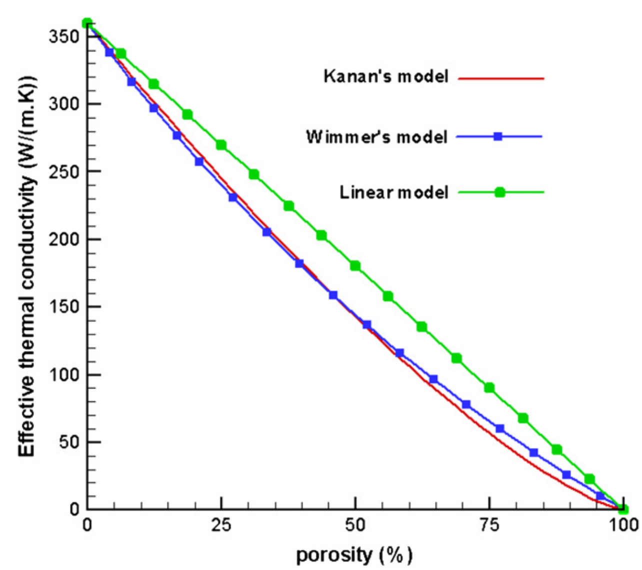


**Fig. 5.** Effective thermal conductivity of the porous granular copper layer versus porosity percentage.

*Thermal contact resistance estimate*: in the multilayer configuration the quality of the thermal contact between two successive layers can be quantified by a single parameter which is the thermal contact resistance. For estimation of this parameter, combination of both experiments and modelling work is needed. In general the value of TCR varies from $10^{-8}$ m$^2$ K W$^{-1}$ (perfect contact) to $10^{-4}$ m$^2$ K W$^{-1}$. Different values within this range were chosen in order to test the model [39–45].

*Heat transfer coefficients estimate*: the thermal convection coefficient on the upper surface of the deposit ($h_1$) depends on the temperature and surface area [46].

$$h_1 = \mathrm{Nu}\cdot k_f/L \tag{26}$$

$L$ is the characteristic length of the specimen ($L \approx 25$ mm), $Nu$ the Nusselt number, and $k_f$ the thermal conductivity of the fluid. $Nu$ is given by [47].

$$\mathrm{Nu} = \left(\sqrt{\mathrm{Nu}_0} + \left[\mathrm{Gr}\cdot\mathrm{Pr}/300\left(\left(1+(0.5/\mathrm{Pr})^{9/16}\right)^{16/9}\right)\right]^{1/6}\right)^2 \tag{27}$$

where $10^{-4} \le Gr\cdot Pr \le 4.10^{14}$, $0.022 \le Pr \le 7.640$, and $Nu_0 = 0.67$ for a plate [48]. $Gr$ and $Pr$ in equation above are Grashof and Prandtl number, respectively defined as:

$$\mathrm{Gr} = gL^3\cdot\rho_f^2\cdot\beta_f\cdot(T_1 - T_{\mathrm{amb}})/\eta_f^2 \text{ and } \mathrm{Pr} = Cp_f\cdot\eta_f/k_f$$

where, $\eta_f$ is the viscosity of the air, and $\beta_f$ the thermal expansion coefficient, $\beta_f = 1/T_f$ for ideal gases.

The effect of the air properties variations with temperature is evaluated at $T_f = 0.5(T_1 + T_{\mathrm{amb}})$. $h_2$ was assumed to be zero.

*Thermophysical data*: the derived model was applied for the case of design of electronic tracks using a moving laser beam treatment of the two-layered material (copper powder deposit on alumina substrate). The averaged thermophysical properties are: air ($k = 46.35\times10^{-3}$ W m$^{-1}$ K$^{-1}$, $\rho = 0.758$ kg m$^{-3}$, $Cp = 1074$ J kg$^{-1}$ K$^{-1}$); copper ($k = 360$ W m$^{-1}$ K$^{-1}$, $\rho = 8954$ kg m$^{-3}$, $Cp = 750$ J kg$^{-1}$ K$^{-1}$); alumina ($k = 20$ W m$^{-1}$ K$^{-1}$, $\rho = 3900$ kg m$^{-3}$, $Cp = 1075$ J kg$^{-1}$ C$^{-1}$); dimensions are: $a_1 = 100\times10^{-6}$ m; and $a_2 = 4\times10^{-3}$ m, convective exchange coefficients are: $h_1 = 15$ W m$^{-2}$ K$^{-1}$; $h_2 = 0.0$ W m$^{-2}$ K$^{-1}$.

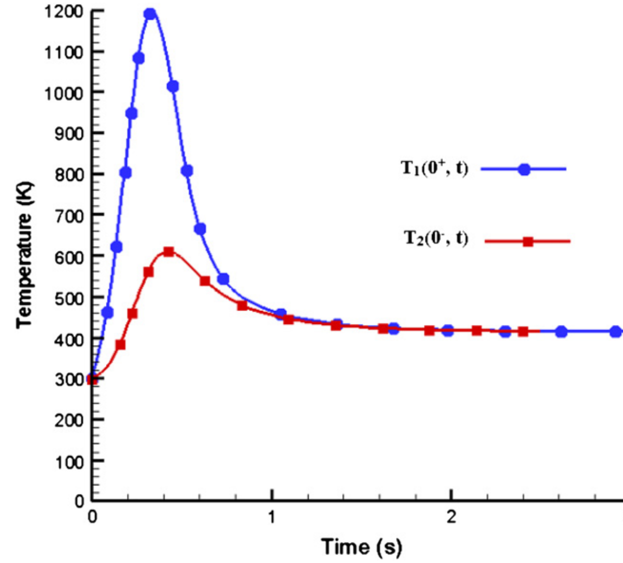


**Fig. 7.** Evolution of temperature $T_1(y = 0^+, t)$ and $T_2(y = 0^-, t)$ besides the interface($y = 0$) versus time, for the two-layred material (copper/alumia), under the following conditions: $TCR = 10^{-4}$ m$^2$ K W$^{-1}$, $Tinit = 300$ °C, $\varphi = 0\%$, $h_1 = 15$ W m$^{-2}$ K$^{-1}$, $V = 2$ mm s$^{-1}$, $P_0 = 10$ W, $r_0 = 0.5$ mm, $A_1 = 0.2$, $A_2 = 0$.

The absorptivity was taken for 1062 μm wavelength, then we set $A_1 = 0.2$ as average value of the absorptivity coefficient for a dense copper, and $A_1 = 0.59$ for copper powder [49].

### 3.2. Illustrative simulations of the full problem

It is first interesting to mention that the present analytical model and numerical ones, is in good agreement with full numerical solution as shown in (Fig. 6).

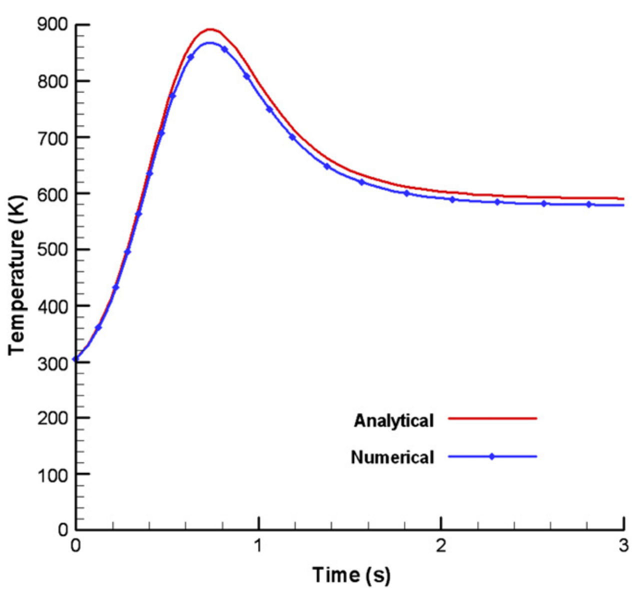


**Fig. 6.** Comparison between analytical and numerical results, of temperature evolution on a point in the upper surface, for the materials copper/alumina: $A_1 = 20\%$, $h_1 = 15$ W m$^{-2}$ K$^{-1}$, $h_2 = 0.0$ W m$^{-2}$ K$^{-1}$ $TCR = 10^{-5}$ m$^2$ K W$^{-1}$, $P = 50$ W, $r_0 = 1$ mm, $V = 2$ m m s$^{-1}$, $Tinit = 300$ °C.

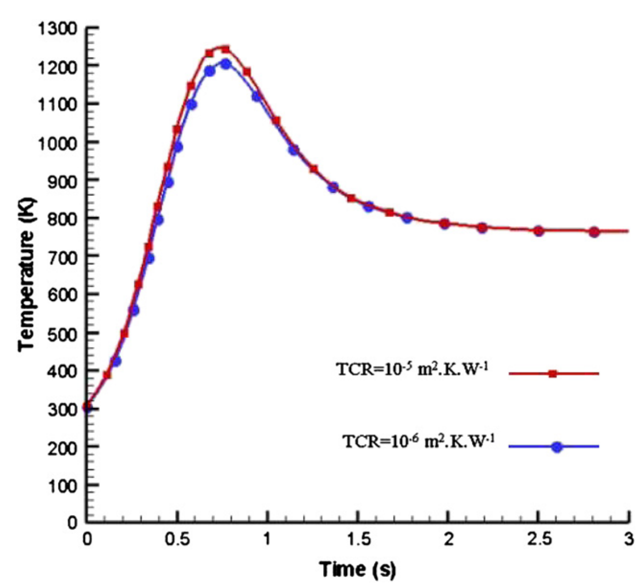


**Fig. 8.** Effect of TCR on temperature $T_1(y = a_1, t)$ on a point in the upper surface versus time, for the two-layered material copper/alumina: $A_1 = 0.2$, $h_1 = 15$ W m$^{-2}$ K$^{-1}$, $h_2 = 0.0$ W m$^{-2}$ K$^{-1}$, $P = 20$ W, $r_0 = 1$ mm, $V = 1$ mm s$^{-1}$, $Tinit = 300$ °C, $\varphi = 0\%$.

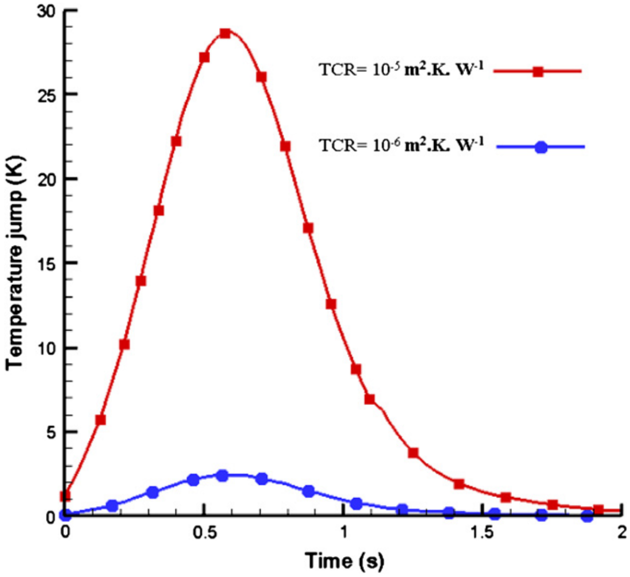


**Fig. 9.** Effect of TCR on temperature gap ($T_1(y=0^+,t)$-$T_2(y=0^-,t)$) besides the interface($y=0$) versus time, for the two-layred material (copper powder/alumia), under the following conditions: $Tinit=300\,°C$, $\varphi=0\%$, $h_1=15\ W\,m^{-2}\,K^{-1}$, $V=1\ mm\,s^{-1}$, $P_0=20\ W$, $r_0=0.5\ mm$, $A_1=0.2$, $A_2=0$.

The variables above when inserted in the temperature expressions derived for the porous case (physical problem sketched in Fig. 4), show the results shown in Figs. 7–11.

Fig. 7 shows the temperature evolution versus time, besides the interface, for a two-layered material

Material (copper powder/alumina), under the following conditions, we can see clearly that under the contact effect (TCR), the temperature jumb increases. So the bad contact prevents the normal diffusion of heat from the upper layer to the substrate.

In Fig. 8 we can see the effect of TCR on the temperature evolution at a chosen monitoring point on the upper surface, so for a weak value of TCR (quasi-perfect contact), then the evacuation of heat flux from the first layer to the substrate is facilitated.

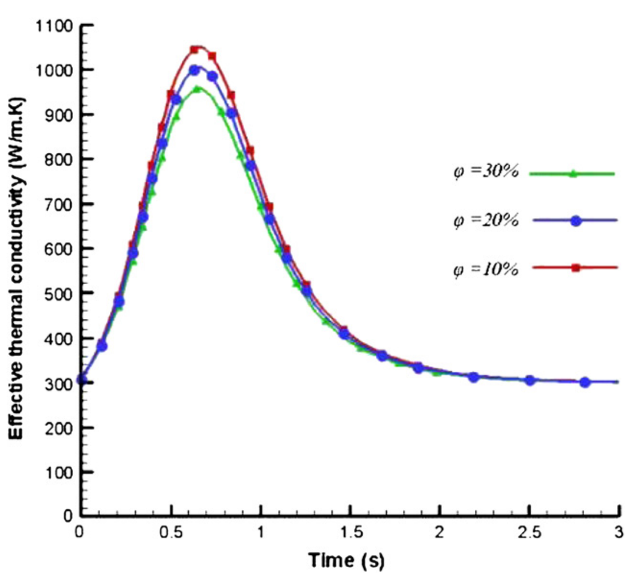


**Fig. 10.** Evolution of Temperature $T_1(a_1, t)$ versus time, for different percentages of the deposit porosity, under the following conditions: $TCR=10^{-6}\ m^2\,K\,W^{-1}$, $h_1=15\ W\,m^{-2}\,K^{-1}$, $Tinit=300\,°C$, $V=1\ mm\,s^{-1}$, $P_0=10\ W$, $r_0=0.5\ mm$, $A_1=0.59$, $A_2=0$.

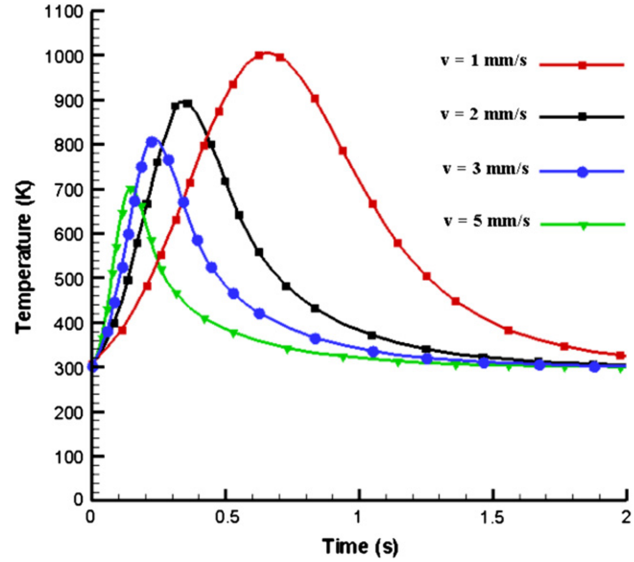


**Fig. 11.** Evolution of Temperature $T_1(a_1, t)$ versus time, for different value of laser beam velocity, under the following conditions: $TCR=10^{-5}\ m^2\,K\,W^{-1}$, $h_1=15\ W\,m^{-2}\,K^{-1}$, $Tinit=300\,°C$, $P_0=10\ W$, $r_0=0.5\ mm$, $A_1=0.59$, $A_2=0$, $\varphi=20\%$.

The temperature gap at the interface ($\Delta T=T_1(0^+,t)-T_2(0^-,t)$) is shown in Fig. 9, so this gap is much higher for important values of TCR, we can see also that, when TCR value is multiplied by a coefficient 10 for example, the gap is ten times higher, so the following relation is verified: $\Delta T=\mathrm{TCR}.\ \Phi$ ($\Phi$ is the heat flux across the interface). The temperature jump is proportional to the value of TCR, for a given constant heat flux.

The effect of porosity percentage on the evolution of temperature for a single point of the upper surface is shown in Fig. 10. It is clear that, as the porosity increases, the effective thermal conductivity and the reached maximum temperature decrease.

Fig. 11 illustrates the temperature evolution of a single point in the upper surface of the deposit, for different values of a scanning velocity ($v=1$, 2, 3 and 5 $mm\,s^{-1}$).

## 4. Conclusion

An analytical solution of transient heat conduction in a two-layered material with imperfect thermal contact subjected to a moving gaussian laser beam; was derived in finite dimensions. The thermal contact resistance at the interface of the two layers was considered. The obtained results are in agreement with those obtained numerically.

The method used is separation of variables, and the convergence of the series is reached for the first twenty eigenvalues, computed numerically by using the Newton–Raphson method.

The porous aspect of the granular deposited layer has been taken into account in this model. The porosity affects the thermal behaviour of the granular deposit.

The methodology has been developed; assumptions and conditions of use were given. By the way, the models mentioned routes for taking into account complex aspect of heat and mass transfers in multilayered material.

This 1D analytical model will be extended to 2D and 3D geometries, and the multi-scans of the workpiece by the laser beam process.

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